Journal of Mathematical Analysis and Applications **243,** 32–46 (2000)
doi:10.1006/jmaa.1999.6645, available online at http://www.idealibrary.com on IDEAL®

A Note on the Synthesis of Nonquadratic Optimal Control in a One-Dimensional Linear System

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Submitted by Leonard D. Berkovitz

Received July 20, 1998

The synthesis problem of the optimal control is investigated in a one-dimensional linear system with a cost function which is Hölderian with respect to the control and convex with respect to the final state. Under some conditions it is proved that the closed-loop synthesis is given by a nonlinear state feedback defined in terms of the solution of a Bernoulli type operator equation. It is also shown that the feedback operator can be explicitly expressed by solutions of associated algebraic equations. © 2000 Academic Press

1. INTRODUCTION

It is well known, cf., e.g., [1, 7], that quadratic optimal control for linear systems can be synthesized in closed-loop form by a linear state feedback which is defined in terms of the solution of a Riccati differential equation. It was also shown in [5, 6] that a similar result holds for a one-dimensional linear system when a Hölderian criterion is substituted to the quadratic cost function, with a Bernoulli differential equation in place of the Riccati equation (see also Example 1 in Section 4 below). Recently the optimal control synthesis problem has been investigated in [10] for a linear system with a criterion which is still quadratic with respect to the control but only smooth convex with respect to the state (see also [8, 9]). There it is proved that the closed-loop synthesis of the optimal control can be realized by a nonlinear state feedback involving the solution of a "quasi-Riccati" operator equation which can be explicitly expressed by solutions of algebraic and integral equations. The purpose of the present paper is to show that a similar fact holds in the case of a one-dimensional linear system when the cost function is Hölderian with respect to the control and convex with

0022-247X/00 $35.00

respect to the final state, with a Bernoulli type operator equation in place of the "quasi-Riccati" equation.

More precisely we deal with an optimal control problem on $[0, T]$, where T is some fixed finite positive number, for a system governed by the linear state equation

$$\frac{dx}{ds} = A(s)x + B(s)u(s), \qquad 0 \le s \le T, \qquad x(0) = \xi. \tag{1}$$

Here the state and control functions x and u and the initial data ξ take values in $\mathbb{R}$. The set of admissible control functions u is chosen as $\mathscr{U} = L^p([0, T]; \mathbb{R})$ for some fixed real number $p > 1$. In what follows q denotes the conjugate value of p, i.e., $q = p/(p-1)$. We assume that

(A1) A and B are continuous functions of $[0, T]$ into $\mathbb{R}$.

Then of course, for each control function $u \in \mathscr{U}$ and arbitrary initial data ξ, Eq. (1) has a uniquely defined solution which shall be called the corresponding trajectory of the system.

We consider the performance criterion defined for a control function u and an initial data ξ by

$$J(u; \xi) = P(x(T)) + \frac{1}{p}\int_0^T R(s)|u(s)|^p \, ds, \tag{2}$$

where x is the corresponding trajectory. Here p, which is the real number involved in the definition of $\mathscr{U}$, may be named the Hölder index of the cost function with respect to the control.

We make the following assumptions:

(A2) P is a convex function of $\mathbb{R}$ into $\mathbb{R}$ which is C^1 with derivative P_x.

(A3) R is a continuous function of $[0, T]$ into $\mathbb{R}$ such that there exists a constant $\delta > 0$ for which $R(s) \ge \delta$ for each $s \in [0, T]$.

In the rest of the paper the control problem $(\mathscr{P}_\xi)$ is investigated,

$$(\mathscr{P}_\xi) \qquad \min_{u \in \mathscr{U}} J(u; \xi), \tag{3}$$

where $J(u; \xi)$ is given by (2). More generally we shall consider the control problem denoted by $(\mathscr{P}_\xi^t)$ which is, for some $t \in [0, T]$, the analogue on $[t, T]$ of problem $(\mathscr{P}_\xi)$. In that case the linear system (1) starts from ξ at time t, the set of admissible control functions is $\mathscr{U}^t = L^p([t, T]; \mathbb{R})$, and the performance criterion $J^t(u; \xi)$ is defined similarly to $J(u; \xi)$ by (2) with an integral on $[t, T]$ only. Of course problems $(\mathscr{P}_\xi)$ and $(\mathscr{P}_\xi^0)$ coincide.

In Section 2 the existence and uniqueness of optimal controls in problems $(\mathscr{P}_\xi^t)$ are proved. Then in Section 3 a necessary condition for optimality is derived and a Bernoulli type operator equation to be solved for the closed-loop synthesis of optimal controls is exhibited. The synthesis problem is treated in Section 4 where it is shown that the solution of the Bernoulli type equation can be explicitly expressed by solutions of algebraic equations. Moreover some examples are discussed.

2. EXISTENCE AND UNIQUENESS

Here we show that the control problems under study are uniquely solvable.

PROPOSITION 1. *Suppose that assumptions* (A1)–(A3) *are satisfied. Then for any* $(t, \xi) \in [0, T] \times \mathbb{R}$, *there exists a unique optimal control* $u_\xi^t \in \mathscr{U}^t$ *for* $(\mathscr{P}_\xi^t)$.

Proof. Obviously we may restrict the proof to the case of $(\mathscr{P}_\xi)$; i.e., $t = 0$. Let u be arbitrary in $\mathscr{U}$. The corresponding trajectory of system (1) is expressed by the formula

$$x(t) = \phi(t,0)\xi + \int_0^t \phi(t,s)B(s)u(s)\,ds, \qquad 0 \le t \le T, \tag{4}$$

where ϕ is the transition function; i.e., $\phi(t,s) = \exp\{\int_s^t A(r)\,dr\}$. Then combining (2) and (4) we get

$$\frac{1}{p}\int_0^T R(t)|u(t)|^p\,dt = J(u;\xi) - P\left(\phi(T,0)\xi + \int_0^T \phi(T,s)B(s)u(s)\,ds\right).$$

Now exploiting assumptions (A1)–(A3) and the fact that for a C^1 convex function f of $\mathbb{R}$ into $\mathbb{R}$ with derivative f_y one has

$$f(y) \ge f(0) + f_y(0)y, \qquad \forall y \in \mathbb{R};$$

it is immediately shown that there exist constants $\alpha(\xi) \in \mathbb{R}$ and $\beta > 0$ such that

$$\frac{\delta}{p}\|u\|_{\mathscr{U}}^p \le \frac{1}{p}\int_0^T R(t)|u(t)|^p\,dt \le J(u;\xi) + \alpha(\xi) + \beta\|u\|_{\mathscr{U}}. \tag{5}$$

But it can be easily checked that

$$|y|^p - p\lambda y + (p-1)|\lambda|^q \ge 0, \qquad \forall(\lambda, y) \in \mathbb{R}^2, \tag{6}$$

where moreover the equality is achieved for

$$y = \operatorname{sgn}(\lambda)|\lambda|^{q/p}. \tag{6$'$}$$

Hence, making use of (6), from (5) we get

$$\begin{aligned} 0 &\le \frac{\delta}{p}\left(\|u\|_{\mathscr{U}}^{p} - p\frac{\beta}{\delta}\|u\|_{\mathscr{U}} + (p-1)\left(\frac{\beta}{\delta}\right)^{q}\right) \\ &\le J(u;\xi) + \left[\alpha(\xi) + q^{-1}\delta\left(\frac{\beta}{\delta}\right)^{q}\right]. \end{aligned} \tag{7}$$

Since inequality (7) holds for any $u \in \mathscr{U}$, the infimum $\inf_{u \in \mathscr{U}} J(u;\xi)$ is finite for a given ξ and a minimizing sequence is bounded so that a weakly convergent subsequence exists in $\mathscr{U}$. On the other hand, for arbitrary u and v in $\mathscr{U}$, from the convexity one may verify that for $\theta \in [0,1]$ we have

$$\begin{aligned} \theta J(u;\xi) + (1-\theta)J(v;\xi) \ge J(\theta u + (1-\theta)v;\xi) \\ + \frac{1}{p}\int_0^T R(t)\Big[\theta|u(t)|^p + (1-\theta)|v(t)|^p \\ - |\theta u(t) + (1-\theta)v(t)|^p\Big]\,dt. \end{aligned}$$

Thus, since the function $y \to |y|^p$ is strictly convex, $J(.;\xi)$ is a strictly convex and strongly continuous functional on $\mathscr{U}$ and hence weakly lower semi-continuous. Therefore, by the basic theorem in variational analysis (cf., e.g., [3]), for any given ξ, the problem $(\mathscr{P}_\xi)$ has a unique optimal control.

3. OPTIMALITY CONDITIONS

At first we give conditions that optimal controls must verify.

PROPOSITION 2. *Suppose that assumptions* (A1)–(A3) *are satisfied and fix* $t \in [0,T]$ *and* $\xi \in \mathbb{R}$. *If* u_ξ^t *is the optimal control in* $\mathscr{U}^t$ *of problem* $(\mathscr{P}_\xi^t)$ *and* x_ξ^t *is the corresponding optimal trajectory, then the equation is verified,*

$$u_\xi^t(s) = -(R(s))^{-q/p}\operatorname{sgn}(B(s))|B(s)|^{q/p}\operatorname{sgn}\big(l_\xi^t(s)\big)\big|l_\xi^t(s)\big|^{q/p}, \quad s \in [t,T], \tag{8}$$

where l_ξ^t is given by

$$l_\xi^t(s) = \phi(T,s)P_x\big(x_\xi^t(T)\big), \qquad s \in [t,T], \tag{9}$$

with

$$x_\xi^t(s) = \phi(s,t)\xi + \int_t^s \phi(s,r)B(r)u_\xi^t(r)\,dr, \qquad s \in [t,T]. \tag{10}$$

Proof. Again we may restrict the proof to the case of $(\mathscr{P}_\xi)$; i.e., $t = 0$. We just apply the Pontryagin maximum principle (cf., e.g., [4]). Here the Hamiltonian function is

$$H(y,v,l,s) = \frac{1}{p}R(s)|v|^p + l(A(s)y + B(s)v).$$

From (6) and (6′) it is easy to see that it achieves its minimum with respect to v at the point

$$v = -(R(s))^{-q/s}\operatorname{sgn}(B(s))|B(s)|^{q/p}\operatorname{sgn}(l)|l|^{q/p}.$$

Therefore if u_ξ^0 is the optimal control in $\mathscr{U}$ of problem $(\mathscr{P}_\xi)$ and x_ξ^0 is the corresponding optimal trajectory, then Eq.(8) must be satisfied for $t = 0$ with l_ξ^0 the solution of the backward differential equation

$$\frac{dl}{ds} = -A(s)l, \qquad 0 \le s \le T, \quad l(T) = P_x\big(x_\xi^0(T)\big).$$

Clearly this means that l_ξ^0 must satisfy (9) for $t = 0$. Since moreover from (1) Eq. (10) holds, the proof is achieved.

Now in order to state a sufficient condition of optimality for a feedback control we introduce the Bernoulli type partial differential equation,

$$\begin{gathered} S_t(t,x) + S_x(t,x)A(t)x \\ - q^{-1}|B(t)|^q(R(t))^{-q/p}|S_x(t,x)|^q = 0, \qquad (t,x) \in [0,T]\times\mathbb{R}, \\ S(T,x) = P(x),\ x \in \mathbb{R}. \end{gathered} \tag{11}$$

A continuous mapping S of $[0,T]\times\mathbb{R}$ into $\mathbb{R}$ is called a solution of the Bernoulli type Eq. (11) if it is continuously differentiable in t and x, respectively, and it satisfies (11) and moreover for any $t \in [0,T]$ and

$\xi \in \mathbb{R}$ the Cauchy problem,

$$\frac{dx}{ds} = A(s)x - |B(s)|^q (R(s))^{-q/p} \operatorname{sgn}(S_x(s,x)) |S_x(s,x)|^{q/p},$$
$$t \le s \le T,\ x(t) = \xi, \quad (12)$$

has a global continuous solution on $[t, T]$. Notice that the properties of S then assure that the solution of (12) is unique.

Now we are able to give a statement which shows the relationship of Eq. (11) with the closed-loop synthesis of the optimal control problem $(\mathscr{P}_\xi^t)$.

PROPOSITION 3. *Suppose that assumptions* (A1)–(A3) *are satisfied. Let S be a solution of the Bernoulli type Eq.* (11) *and fix $t \in [0, T]$ and $\xi \in \mathbb{R}$. Let x_ξ^t be the solution of Eq.* (12). *Then x_ξ^t is the optimal trajectory of system* (1) *in the optimal control problem* $(\mathscr{P}_\xi^t)$. *In other words, if u_ξ^t is the state feedback control defined for $s \in [t, T]$ by*

$$u_\xi^t(s) = -(R(s))^{-q/p} \operatorname{sgn}(B(s)) |B(s)|^{q/p} \times \operatorname{sgn}\Big(S_x\big(s, x_\xi^t(s)\big)\Big) \Big|S_x\big(s, x_\xi^t(s)\big)\Big|^{q/p}, \quad (13)$$

then u_ξ^t is the optimal control of problem $(\mathscr{P}_\xi^t)$. *Moreover the number $S(t, \xi)$ is the minimum value of the performance criterion $J^t(.;\xi)$ in problem* $(\mathscr{P}_\xi^t)$; *i.e.,* $S(t,\xi) = J^t(u_\xi^t; \xi)$.

Proof. At first observe that from (6) and (6′) the quantity

$$\frac{1}{p} R(s)|v|^p + S_y(s,y) B(s) v$$

achieves its minimum with respect to v at the point

$$v = -(R(s))^{-q/p} \operatorname{sgn}(B(s)) |B(s)|^{q/p} \operatorname{sgn}\big(S_y(s,y)\big) \big|S_y(s,y)\big|^{q/p}$$

and that the value of the minimum is

$$-q^{-1}(R(s))^{-q/p} |B(s)|^q \big|S_y(s,y)\big|^q.$$

Therefore, according to the Bellman principle of optimality (cf., e.g., [4]), Eq. (11) is nothing but the so-called Bellman or Hamilton–Jacobi equation associated to problems $(\mathscr{P}_\xi^t)$.

Here again we restrict the proof of the assertions to the case of $(\mathscr{P}_\xi)$, i.e., we take $t = 0$. Let $u \in \mathscr{U}$ be any admissible control and let x be the corresponding trajectory of system (1) starting from ξ at time 0. Then from (11) we have

$$S_s(s, x(s)) + S_y(s, x(s)) A(s) x(s) + S_y(s, x(s)) B(s) u(s) + \frac{1}{p} R(s) |u(s)|^p \geq 0.$$

Thus, taking (1) into account, it follows that

$$\frac{d}{ds} S_s(s, x(s)) = S_s(s, x(s)) + S_y(s, x(s)) \frac{dx}{ds}(s) \geq -\frac{1}{p} R(s) |u(s)|^p.$$

Hence, integrating from 0 to T, we get

$$S(T, x(T)) - S(0, x(0)) \geq -\frac{1}{p} \int_0^T R(s) |u(s)|^p \, ds,$$

and, since $x(0) = \xi$ and $S(T, y) = P(y)$, also

$$S(0, \xi) \leq P(x(T)) + \frac{1}{p} \int_0^T R(s) |u(s)|^p \, ds = J(u; \xi). \tag{14}$$

Now for u_ξ^0 defined by (13) for $t = 0$ with x_ξ^0 the solution of (12) we have

$$B(s) u_\xi^0(s) = -|B(s)|^q (R(s))^{-q/p} \operatorname{sgn}\left(S_y\left(s, x_\xi^0(s)\right)\right) \left|S_y\left(s, x_\xi^0(s)\right)\right|^{q/p}.$$

Therefore, due to uniqueness, x_ξ^0 is nothing but the trajectory of system (1) corresponding to the feedback control u_ξ^0 which is admissible in $\mathscr{U}$.

With (u_ξ^0, x_ξ^0) substituted to (u, x), calculations similar to those above which have led to (14) apply with equality throughout in place of inequality to give

$$S(0, \xi) = J\left(u_\xi^0; \xi\right). \tag{15}$$

But (14) and (15) say exactly that u_ξ^0 is optimal and that $S(0, \xi)$ is the minimum value of the performance criterion in problem $(\mathscr{P}_\xi)$.

Remark 1. (a) Of course Proposition 3 says that if a solution S of the Bernoulli type Eq. (11) can be found then the closed-loop synthesis of optimal controls in problems $(\mathscr{P}_\xi^t)$ is given by the state feedback operator F defined for $(s, x) \in [0, T] \times \mathbb{R}$ by

$$F(s, x) = -(R(s))^{-q/p} \operatorname{sgn}(B(s)) |B(s)|^{q/p} \operatorname{sgn}\left(S_y(s, x)\right) \left|S_y(s, x)\right|^{q/p}.$$

(b) Let us point out that the Bernoulli type Eq. (11) is not the exact analogue of the "quasi-Riccati" equation used in [10]. Indeed here, due to the last assertion in Proposition 3, the partial differential equation concerns the so-called value function of the control problem. But in [10] the partial differential equation concerns the partial derivative of the value function with respect to the space variable.

4. CLOSED-LOOP SYNTHESIS

In this section we study the closed-loop synthesis problem on the basis of optimality conditions exhibited in Section 3. At first we start with an example which illustrates the solvability of the problem.

EXAMPLE 1. Here we consider the particular case where the cost function is also Hölderian with index p with respect to the final state of the system. This means that we take $P(y) = (1/p)P|y|^p$ for all $y \in \mathbb{R}$ where P is a given nonnegative number. Let us look for a solution of (11) in the form $S(t, y) = (1/p)S(t)|y|^p$ where S is some derivable function. Since for such an S we have

$$S_s(s, y) = \frac{1}{p}\dot{S}(s)|y|^p, \qquad S_y(s, y) = S(s)\operatorname{sgn}(y)|y|^{p-1},$$

from (11) we get the equation

$$\begin{gathered}\dot{S}(s)|y|^p + pA(s)S(s)|y|^p \\ -(p-1)(R(s))^{-q/p}|B(s)|^q|S(s)|^q|y|^p = 0, \qquad (s, y) \in [0, T] \times \mathbb{R}, \\ S(T)|y|^p = P|y|^p, \qquad y \in \mathbb{R},\end{gathered}$$

or equivalently

$$\dot{S}(s) + pA(s)S(s) - (p-1)(R(s))^{-q/p}|B(s)|^q|S(s)|^q = 0, \qquad S(T) = P. \tag{16}$$

But the Bernoulli differential Eq. (16) has a unique nonnegative continuous solution S which is given for $s \in [0, T]$ by

$$S(s) = \left(P^{-q/p} + \int_s^T (R(s))^{-q/p}(|B(r)|\phi(T, r))^q\, dr\right)^{1-p} (\phi(T, s))^p.$$

It is easy to verify that with $S(s)\operatorname{sgn}(y)|y|^{p-1}$ in place of $S_y(s, y)$, Eq. (12) becomes

$$\frac{dx}{ds} = \left[A(s) - |B(s)|^q\big((R(s))^{-1}S(s)\big)^{q/p}\right]x, \qquad t \le s \le T,\ x(t) = \xi,$$

which has a global solution x_ξ^t on $[t, T]$. Then we may apply Proposition 3 and Remark 1(a) to get that here the closed-loop synthesis of optimal controls in problems $(\mathscr{P}_\xi^t)$ is given by the linear state feedback operator F defined for $(s, x) \in [0, T] \times \mathbb{R}$ by

$$F(s,x) = -\operatorname{sgn}(B(s)) \frac{\left((R(s))^{-1}|B(s)|\right)^{q/p}(\phi(T,s))^q}{P^{-q/p} + \int_s^T (R(r))^{-q/p}(|B(r)|\phi(T,r))^q \, dr} x.$$

Of course for $p = 2$ (and then also $q = 2$), the result reduces to the well-known result in the quadratic case where Eq. (16) reduces to a Riccati differential equation.

Now we investigate the case of a more general term P in the cost function. Let u_ξ^t be the optimal control in problem $(\mathscr{P}_\xi^t)$ and let x_ξ^t be the corresponding optimal trajectory. Then from Proposition 2, by (8) and (9) we get for $r \in [t, T]$

$$u_\xi^t(r) = -(R(r))^{-q/p} \operatorname{sgn}(B(r))(|B(r)|\phi(T,r))^{q/p}$$
$$\times \operatorname{sgn} P_x\left(x_\xi^t(T)\right)\left|P_x\left(x_\xi^t(T)\right)\right|^{q/p}.$$

Hence, inserting this into (10) for $s = T$, it comes that

$$x_\xi^t(T) = \phi(T,t)\xi - \left[\int_t^T (R(r))^{-q/p}(|B(r)|\phi(T,r))^q \, dr\right]$$
$$\times \operatorname{sgn}\left(P_x\left(x_\xi^t(T)\right)\right)\left|P_x\left(x_\xi^t(T)\right)\right|^{q/p}.$$

In other words, $x_\xi^t(T)$ appears for $x = \xi$ as a solution of the algebraic equation in the variable $y \in \mathbb{R}$,

$$y + \Lambda(t)Q(y) = \phi(T,t)x, \tag{17}$$

where Λ is the positive function defined by

$$\Lambda(t) = \int_t^T (R(r))^{-q/p}(|B(r)|\phi(T,r))^q \, dr, \qquad t \in [0,T] \tag{18}$$

and Q is the function given by

$$Q(x) = \operatorname{sgn}(P_x(x))|P_x(x)|^{q/p}, \qquad x \in \mathbb{R}. \tag{19}$$

Moreover, from the definition of $J^t(.;\xi)$ we can also write

$$J^t\left(u_\xi^t;\xi\right) = P\left(x_\xi^t(T)\right) + \frac{1}{p}\Lambda(t)\left|Q\left(x_\xi^t(T)\right)\right|^p.$$

Then, taking into account the last assertion in Proposition 3, provided that for all $(t, x) \in [0, T] \times \mathbb{R}$ Eq. (17) has a unique solution $y = H(t, x)$, we get the following function S as a candidate to solve the Bernoulli type Eq. (11):

$$S(t, x) = P(H(t, x)) + \frac{1}{p}\Lambda(t)\left|Q(H(t, x))\right|^p. \tag{20}$$

So we investigate uniqueness and differentiability of the solution of the synthesis Eq. (17). We need to introduce the following complementary assumption:

(A4) Q defined by (19) is a C^1 function with derivative Q_x.

Let us mention that typical examples of functions P satisfying (A1)–(A4) are $P(x) = |x|^\alpha$ for any fixed $\alpha \geq p$. Notice that the convexity of P, assumed by (A1), implies that the function Q is nondecreasing so that if (A4) is fulfilled then also $Q_x \geq 0$. Moreover by definition (19) we have

$$\operatorname{sgn}(Q(x)) = \operatorname{sgn}(P_x(x)), \qquad |Q(x)|^p = |P_x(x)|^q. \tag{21}$$

LEMMA 1. *Suppose that assumptions* (A1)–(A4) *are satisfied. Then for any* $(t, x) \in [0, T] \times \mathbb{R}$ *there exists a unique solution* $y \in \mathbb{R}$, *denoted by* $y = H(t, x)$, *of Eq.* (17), *where* Λ *and* Q *are the functions defined by* (18) *and* (19), *respectively. The mapping* H *is continuously differentiable in* (t, x) *and its partial derivatives are*

$$H_x(t, x) = \left(1 + \Lambda(t) Q_x(H(t, x))\right)^{-1} \phi(T, t) \tag{22}$$

and

$$H_t(t, x) = \frac{(R(t))^{-q/p}\left(|B(t)|\phi(T, t)\right)^q Q(H(t, x)) - A(t)\phi(T, t)x}{1 + \Lambda(t) Q_x(H(t, x))}. \tag{23}$$

Proof. From the discussion above, existence is already proved. So we have only to prove uniqueness and differentiability. Define a mapping

$$C(t, x, y) = y + \Lambda(t) Q(y) - \phi(T, t)x.$$

Since Λ and Q_y are nonnegative, the derivative

$$C_y(t, x, y) = 1 + \Lambda(t) Q_y(y)$$

is boundedly invertible. Hence by the implicit function theorem (cf., e.g., [2]) we conclude that the solution of Eq. (17) is unique and differentiable in variables t and x. Moreover

$$H_x(t,x) = -\frac{C_x(t,x,H(t,x))}{C_y(t,x,H(t,x))}, \qquad H_t(t,x) = -\frac{C_t(t,x,H(t,x))}{C_y(t,x,H(t,x))}.$$

Computing derivatives C_y, C_x, and C_t and substituting their expressions into the above equalities one easily gets (22) and (23).

Now we are able to state our result about the synthesis problem.

PROPOSITION 4. *Suppose that assumptions* (A1)–(A4) *are satisfied. Define the mapping S on* $[0,T] \times \mathbb{R}$ *by Eq.* (20) *where H is the mapping shown in Lemma* 1. *Then S is a solution of the Bernoulli type Eq.* (11). *Moreover the following identity holds,*

$$S_x(t,x) = P_x(H(t,x))\phi(T,t), \qquad (t,x) \in [0,T] \times \mathbb{R}. \tag{24}$$

Proof. Since for Λ given by (18) one has $\Lambda(T) = 0$, is obvious from (17) that $H(T,x) = x$ and then also from (20) that $S(T,x) = P(x)$.
From (20) and (21) we obtain

$$\begin{aligned} S_x(t,x) &= \Big[P_x(H(t,x)) + \Lambda(t)\mathrm{sgn}(Q(H(t,x)) \\ &\qquad \times |Q(H(t,x))|^{p-1} Q_x(H(t,x))\Big] H_x(t,x) \\ &= [1 + \Lambda(t)Q_x(H(t,x))] P_x(H(t,x)) H_x(t,x). \end{aligned} \tag{25}$$

Hence, making use of (22), we see that Eq. (24) holds.
Similarly, from (18), we get

$$\begin{aligned} S_t(t,x) &= [1 + \Lambda(t)Q_x(H(t,x))] P_x(H(t,x)) H_t(t,x) \\ &\quad - \frac{1}{p}(R(t))^{-q/p}(|B(t)|\phi(T,t))^q |Q(H(t,x))|^p. \end{aligned} \tag{26}$$

Now, from (25) and (26), we may write

$$\begin{aligned} &S_t(t,x) + S_x(t,x)A(t)x \\ &\quad = [1 + \Lambda(t)Q_x(H(t,x))] P_x(H(t,x)) [H_t(t,x) + H_x(t,x)A(t)x] \\ &\qquad - \frac{1}{p}(R(t))^{-q/p}(|B(t)|\phi(T,t))^q |Q(H(t,x))|^p. \end{aligned} \tag{27}$$

But, combining (22) and (23), we have

$$H_t(t,x) + H_x(t,x)A(t)x = \frac{(R(t))^{-q/p}(|B(t)|\phi(T,t))^q Q(H(t,x))}{1 + \Lambda(t)Q_x(H(t,x))}. \tag{28}$$

Therefore, by (27) and (28) we may rewrite

$$\begin{aligned} S_t(t,x) &+ S_x(t,x)A(t)x \\ &= (R(t))^{-q/p}(|B(t)|\phi(T,t))^q Q(H(t,x))P_x(H(t,x)) \\ &\quad + \frac{1}{p}(R(t))^{-q/p}(|B(t)|\phi(T,t))^q |Q(H(t,x))|^p. \end{aligned}$$

Again, making use of (19) and (21), it follows that

$$\begin{aligned} S_t(t,x) &+ S_x(t,x)A(t)x \\ &= q^{-1}(R(t))^{-q/p}(|B(t)|\phi(T,t))^q |P_x(H(t,x))|^q. \end{aligned}$$

Consequently, taking (24) into account, we get

$$S_t(t,x) + S_x(t,x)A(t)x = q^{-1}(R(t))^{-q/p}|B(t)S_x(t,x)|^q,$$

what means that Eq. (11) is verified.

Finally we prove the existence of a global solution to the Cauchy problem (12). In fact, in view of the uniqueness just proved, the optimal trajectory x_ξ^t of problem $(\mathscr{P}_\xi^t)$ satisfies

$$x_\xi^t(s) = H\big(s, x_\xi^t(T)\big), \qquad s \in [t,T].$$

Then, by Proposition 2, we see that this x_ξ^t is a global solution of (12).

Remark 2. (a) Due to Remark 1(a), we can interpret Proposition 4 in the sense that the closed-loop synthesis of optimal controls in problems $(\mathscr{P}_\xi^t)$ is given by the state feedback operator F defined for $(t,x) \in [0,T] \times \mathbb{R}$ by

$$\begin{aligned} F(t,x) = -(R(t))^{-q/p} &\operatorname{sgn}(B(t))|B(t)|^{q/p} \\ &\times \operatorname{sgn}\big(P_x(H(t,x))\big)|P_x(H(t,x))\phi(T,t)|^{q/p}. \end{aligned}$$

(b) Notice that when $p = q = 2$, the result reduces to Theorem 4.2 in [10]. Indeed in that case assumptions (A1) and (A4) require nothing but that P is a C^2 and convex function.

(c) In the case where $P(x) = \frac{1}{p}P|x|^p$ with $P \geq 0$, we have $P_x(x) = P\,\mathrm{sgn}(x)|x|^{p-1}$. Hence $Q(x) = P^{q/p}x$, and the synthesis Eq. (17), which becomes

$$y + P^{q/p}\Lambda(t)y = \phi(T,t)x,$$

has the unique solution

$$H(t,x) = \left[1 + P^{q/p}\Lambda(t)\right]^{-1}\phi(T,t)x.$$

Therefore Proposition 4 can be applied and, from Remark 2(a), it is easy to verify that one finds again the solution of the synthesis problem obtained in Example 1 above.

Now let us investigate some complementary examples where the solution of the synthesis problem can be explicitly expressed.

EXAMPLE 2. Here we take $P(x) = P|x|^{2p-1}$ with $P \geq 0$. We have $P_x(x) = (2p-1)P\,\mathrm{sgn}(x)|x|^{2(p-1)}$. Hence $Q(x) = ((2p-1)P)^{q/p}\,\mathrm{sgn}(x)x^2$ and the synthesis Eq. (17) becomes

$$y + ((2p-1)P)^{q/p}\Lambda(t)\mathrm{sgn}(y)y^2 = \phi(T,t)x.$$

Therefore Proposition 4 can be applied and it is readily seen here that $H(t,x) = x$ for $t = T$ and

$$H(t,x) = \mathrm{sgn}(x)\frac{\left[1 + 4((2p-1)P)^{q/p}\Lambda(t)\phi(T,t)|x|\right]^{1/2} - 1}{2((2p-1)P)^{q/p}\Lambda(t)}$$

for $t \in [0,T)$.

EXAMPLE 3. Here we take $P(x) = P|x|^{3p-2}$ with $P \geq 0$. We have $P_x(x) = (3p-2)P\,\mathrm{sgn}(x)|x|^{3(p-1)}$. Hence $Q(x) = ((3p-2)P)^{q/p}x^3$ and the synthesis Eq. (17) becomes

$$y + ((3p-2)P)^{q/p}\Lambda(t)y^3 = \phi(T,t)x.$$

Therefore again Proposition 4 can be applied. Moreover, according to Cardan's formula, it is readily seen that the unique solution of the synthesis equation is $H(t,x) = x$ for $t = T$ and

$$H(t,x) = \left[(\Delta(t,x))^{1/2} + \frac{\phi(T,t)x}{2\beta(t)}\right]^{1/3} - \left[(\Delta(t,x))^{1/2} - \frac{\phi(T,t)x}{2\beta(t)}\right]^{1/3}$$

for $t \in [0, T)$, where

$$\Delta(t, x) = \left[\frac{\phi(T, t)x}{2\beta(t)}\right]^2 + \left[\frac{1}{3\beta(t)}\right]^3$$

with

$$\beta(t) = ((3p - 2)P)^{q/p}\Lambda(t).$$

Observe that taking $p = q = 2$ here, we again find the result of the example in Section 7 of [10].

5. CONCLUDING COMMENTS

In the present paper the problem of open-loop synthesis of optimal control has been completely solved for a one-dimensional linear system with a cost function which is Hölderian with respect to the control and convex with respect to the final state. In the particular situation where the Hölder index is equal to 2, our result reduces to that of [10] concerning a cost function which remains quadratic with respect to the control. It seems that for a one-dimensional system, the extension of our developments to the case of cost functions containing a term involving the current state is feasible so that the parallel with [10] could be completed. But, probably, possible extensions to multidimensional systems will require more work. These questions will be investigated in a forthcoming study.

ACKNOWLEDGMENT

We thank a referee for pointing out some misprints and suggesting some necessary improvements of the first version of this paper.

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